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An effective approach to the problem of time

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Abstract

A practical way to deal with the problem of time in quantum cosmology and quantum gravity is proposed. The main tool is effective equations, which mainly restrict explicit considerations to semiclassical regimes but have the crucial advantage of allowing the consistent use of local internal times in non-deparameterizable systems. Different local internal times are related merely by gauge transformations, thereby enabling relational evolution through turning points of non-global internal times. The main consequence of the local nature of internal time is the necessity of its complex-valuedness, reminiscent of but more general than non-unitarity of evolution defined for finite ranges of time. By several general arguments, the consistency of this setting is demonstrated. Finally, we attempt an outlook on the nature of time in highly quantum regimes. The focus of this note is on conceptual issues.

1 Introduction

The problem of time [1, 2, 3, 4] arises in quantum gravity because the dynamics of a generally covariant theory is fully constrained, without a true Hamiltonian generating evolution with respect to a distinguished time. Moreover, the relational interpretation of evolution is complicated by the *global time problem*, the fact that a globally valid choice of internal time is difficult to find and may not exist. For specific matter systems, such as a free massless scalar field or pressureless dust, deparameterizations with a matter clock can be performed, but these models seem rather special. In order to evaluate the dynamics of quantum gravity and derive potentially observable information from first principles, the problem of time must be overcome at a general level without requiring specific adaptations.

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For most applications of quantum gravity related to potential observables, semiclassical evolution is sufficient or at least provides a great deal of information. One may then hope that such a situation makes tackling the problem of time more feasible since this problem does not play a handicapping role classically; or at least a dedicated analysis of semiclassical evolution should provide insights which may help in attacking the problem in full generality. In this article, we use the effective approach to quantum constraints for finite dimensional systems developed in [5, 6] in the context of the problem of time, and we propose a practical solution employing local rather than global internal times.

The concept of using an internal time even if it is not well-defined globally and switching to a new time when this becomes necessary, is easy to apply classically and very reminiscent of the concept of local coordinates on a manifold. Obviously, if an atlas of local internal times can be made to work even in quantum gravity, this concept has the potential of leading to a solution to the problem of time. From the point of view of state evolution, local internal times in quantum systems have been discussed, for instance, in [7], but no clear solution as regards evolution has been reached. (Specific constructions in the Bianchi I model have been suggested in [8].) The main problem is not unexpected: If time is defined only for a finite range, unitary evolution of states cannot be realized. While classical evolution with respect to a local internal time is unproblematic all the way to and — by patching — even through its turning point, non-unitary quantum evolution is in danger of producing meaningless results long before the end of one local internal time is reached. Moreover, it is not clear how to define quantum observables in such a situation. The technical Hilbert-space issues related to these conceptual problems in the context of time and evolution seem too difficult to be resolved even in simple models, let alone in a practical manner for generic situations in quantum gravity.

At this stage, effective techniques which describe a quantum system and its dynamics via expectation values and moments assigned by a state become important. While these tools describe the full quantum system — usually approximately and for specific classes of states — for many purposes they produce equations that can be treated by well-known classical procedures. As we will see, the new viewpoint also sheds light on issues of time and especially the use of local internal times in quantum systems. Instead of non-unitarity of the evolution, we will encounter the need to use complex-valued times; but in contrast to problems with evolution of states in a Hilbert space, the effective evolution of expectation values and moments with respect to complex time can easily be made sense of. These features of complex time in the effective formulation, as well as analogous ones that lie more hidden in standard treatments of state evolution, are the focus of this note and, in particular, of Section 3. Moreover, switching local internal times within the effective treatment can be handled consistently and requires nothing more than a gauge transformation. Toward the end of this article, we will dare an outlook on the nature of time in non-semiclassical, highly quantum states.

2 Effective constraints

We consider a quantum system subject to a single constraint operator $\hat{C}$ playing the role of a Hamiltonian constraint. Physical states thus satisfy $\hat{C}|\psi\rangle = 0$. Assumptions about the spectrum of $\hat{C}$ will not be made; in particular, effective techniques work for zero in the discrete as well as the continuous part of the spectrum of constraint operators.

For a systematic derivation of effective descriptions for canonical quantum theories we

parameterize states by expectation values and moments rather than wave functions or density matrices: For several pairs of canonical degrees of freedom $(q_1, p_1; q_2, p_2; \ldots; q_n, p_n)$, we use the expectation values $\langle\hat{q}_i\rangle$ and $\langle\hat{p}_i\rangle$, $i = 1, \ldots n$, together with the moments

$$\Delta(q_1^{a_1} p_1^{b_1} q_2^{a_2} p_2^{b_2} \ldots) := \langle(\hat{q}_1 - \langle\hat{q}_1\rangle)^{a_1}(\hat{p_1} - \langle\hat{p_1}\rangle)^{b_1} \ldots\rangle_{\text{Weyl}}$$

(ordered totally symmetrically and defined for $\sum_i (a_i + b_i) \geq 2$) as a complete description of states. (For instance, $\Delta(q_i^2) = (\Delta q_i)^2$ is the position fluctuation of the i-th coordinate.)

The manifold spanned by expectation values and moments carries a phase-space structure defined by the Poisson bracket

$$\{\langle\hat{A}\rangle, \langle\hat{B}\rangle\} = \frac{\langle[\hat{A}, \hat{B}]\rangle}{i\hbar}$$

for any pair of operators $\hat{A}$ and $\hat{B}$, extended to the moments using the Leibniz rule and linearity. If there is a true Hamiltonian, it follows from the Heisenberg equation that the Hamiltonian flow of expectation values and moments is generated by the quantum Hamiltonian $H_Q(\langle\hat{q}\rangle, \langle\hat{p}\rangle, \Delta(\cdots)) = \langle\hat{H}\rangle$.

For a constraint, the expectation values and moments assigned by physical states must satisfy $\langle\hat{C}\rangle = 0$ as a constraint function on the quantum phase space, but also

$$C_{\text{pol}} := \langle(\widehat{\text{pol}} - \langle\widehat{\text{pol}}\rangle)\hat{C}\rangle = 0$$

for all polynomials $\widehat{\text{pol}}$ in basic operators must vanish. This set forms infinitely many first-class constraints for infinitely many variables. Notice the ordering: while moments are defined via a totally symmetric ordering of operators, this is not done for the quantum constraints; otherwise they would not be first class [5]. As a consequence, some of the quantum constraints take complex values. As already shown for deparameterizable systems [5, 6], this complex-valuedness is not problematic. It simply reflects the fact that quantum constraints are formulated on the states that take values on the full algebra of kinematical operators, not all of which correspond to physical observables once the constraint is implemented. Hence, the kinematical moments that appear in the expressions of constraints need not be restricted to real values. After implementing the constraints, reality conditions can be imposed on the physical expectation values and moments — the Dirac observables of the constrained system — and contact with the physical Hilbert space can be made.

The set of infinitely many constraints for infinitely many variables is directly tractable by exact means if the constraints decouple into finite sets, a situation realized only for constraints linear in canonical variables. For more interesting cases one must use approximations that allow one to reduce the system by ignoring subdominant terms. The prime example for such an approximation is the semiclassical expansion, corresponding to states whose moments of high orders are suppressed compared to expectation values and lower-order moments. The simple and still rather general assumption $\Delta(q^a p^b) = O(\hbar^{(a+b)/2})$ allows one to arrange all contributions to the constraints in such a way that to any given finite order in $\hbar$ only finitely many constraints contribute, allowing one to solve for all physical moments up to the order considered. This semiclassicality assumption will be used in the following discussions (except in parts of section 4). We note that this restriction on the states considered still leaves a large class, much larger, certainly, than the common specification of a Gaussian state (whose moments are completely fixed by specifying just the second-order moments) would allow.

2.1 Example: "Relativistic" harmonic oscillator

Consider $\hat{C} = \hat{p}_t^2 - \hat{p}_\alpha^2 - \hat{\alpha}^2$. To second order in the moments, we obtain the quantum constraints [9]

$$\begin{aligned}
C &= \langle\hat{p}_t\rangle^2 - \langle\hat{p}_\alpha\rangle^2 - \langle\hat{\alpha}\rangle^2 + (\Delta p_t)^2 - (\Delta p_\alpha)^2 - (\Delta\alpha)^2 & (1)\\
C_t &= 2\langle\hat{p}_t\rangle\Delta(tp_t) + i\hbar\langle\hat{p}_t\rangle - 2\langle\hat{p}_\alpha\rangle\Delta(tp_\alpha) - 2\langle\hat{\alpha}\rangle\Delta(t\alpha) & (2)\\
C_{p_t} &= 2\langle\hat{p}_t\rangle(\Delta p_t)^2 - 2\langle\hat{p}_\alpha\rangle\Delta(p_tp_\alpha) - 2\langle\hat{\alpha}\rangle\Delta(p_t\alpha) & (3)\\
C_\alpha &= 2\langle\hat{p}_t\rangle\Delta(p_t\alpha) - 2\langle\hat{p}_\alpha\rangle\Delta(\alpha p_\alpha) - i\hbar\langle\hat{p}_\alpha\rangle - 2\langle\hat{\alpha}\rangle(\Delta\alpha)^2 & (4)\\
C_{p_\alpha} &= 2\langle\hat{p}_t\rangle\Delta(p_tp_\alpha) - 2\langle\hat{p}_\alpha\rangle(\Delta p_\alpha)^2 - 2\langle\hat{\alpha}\rangle\Delta(\alpha p_\alpha) + i\hbar\langle\hat{\alpha}\rangle\,. & (5)
\end{aligned}$$

These constraint functions are first-class to order $\hbar$ and therefore generate gauge transformations. This is a key difference between the standard Dirac constraint quantization at the Hilbert-space level and the effective approach: after solving the quantum constraint in the former method all gauge flows are absent in the physical Hilbert space, whereas solving the constraints at the effective level does not immediately lead to gauge invariance. One way of understanding this difference is to note that the states of the physical Hilbert space assign expectation values only to the physical Dirac observables, while in the effective approach, states assign expectation values to all kinematical variables, which in general are subject to gauge even classically. Gauge invariance at the effective level is only achieved by constructing effective Dirac observables, at which point the number of (true) degrees of freedom in the two approaches coincides.

Following [5, 6] we fix the gauge that for deparameterizable systems corresponds to the evolution of $\hat{\alpha}$ and $\hat{p}_\alpha$ in $\hat{t}$, by setting fluctuations of the latter to zero

$$(\Delta t)^2 = \Delta(t\alpha) = \Delta(tp_\alpha) = 0\,. \qquad (6)$$

Imaginary contributions to the constraints arise, which require some of the moments to take complex values. For instance, $\Delta(tp_t) = -\frac{1}{2}i\hbar$ if one imposes the above gauge choice. All the gauge-fixed moments refer to t which, when chosen as time in this deparameterizable system, is not represented as an operator and does not generate physical moments. The gauge-dependence or complex-valuedness of these moments is, therefore, not a problem. In fact, the complex-valuedness of the moments guarantees that generalized uncertainty relations are respected even if some fluctuations vanish. In particular, the gauge (6) actually leads to a saturation of the (generalized) uncertainty relation $(\Delta t)^2(\Delta p_t)^2 - (\Delta(tp_t))^2 \geq \hbar^2/4$.

Moments not involving time or its momentum, on the other hand, should have a physical analog taking strictly real values. That this is the case can be inferred from the fact that the second-order effective quantum system can be deparameterized by solving for $\langle\hat{p}_t\rangle = \pm H_Q$ with the quantum Hamiltonian

$$H_Q = \sqrt{\langle\hat{p}_\alpha\rangle^2 + \langle\hat{\alpha}\rangle^2}\left(1 + \frac{\langle\hat{\alpha}\rangle^2(\Delta p_\alpha)^2 - 2\langle\hat{\alpha}\rangle\langle\hat{p}_\alpha\rangle\Delta(\alpha p_\alpha) + \langle\hat{p}_\alpha\rangle^2(\Delta\alpha)^2}{2(\langle\hat{p}_\alpha\rangle^2 + \langle\hat{\alpha}\rangle^2)^2}\right).$$

Solving the Hamiltonian equations of motion for $\langle\hat{\alpha}\rangle(t)$, $\langle\hat{p}_\alpha\rangle(t)$, $\Delta(\cdots)(t)$ gives the Dirac observables of the constrained system, on which reality can easily be imposed simply by requiring real initial values at some t. Although there is a true operator $\hat{t}$ at the kinematical level, its expectation value does not appear in the effective constraints. In the final equations of motion for the physical evolving observables, it just appears as an evolution parameter.

2.2 Non-deparameterizable systems

Zeit ist das, was man an der Uhr abliest. (Time is what you read off the clock.)

Albert Einstein

We now turn to systems in which no global internal time exists, realized, for instance, in the presence of a time-dependent potential in the constraint operator. To be specific, by a non-global internal time we mean a clock variable whose equal-time surfaces may be intersected more than once or not at all by a classical trajectory; the clock will, therefore, encounter one or more extrema in the course of classical evolution. Such systems do occur in the context of general relativity, one simple example being a $k = 1$ FRW universe filled with a massive scalar field. While classically there is no profound problem with non-global clocks since, in principle, one can always revert to a time coordinate, i.e., the gauge parameter along the flow of the Hamiltonian constraint, the time coordinate is absent in the quantum theory, where physical states are automatically gauge invariant. In these situations evolution with respect to local internal times is required. A coherent state of the corresponding quantum system which is peaked on a classical trajectory, must then decay beyond the classical turning point of the local clock, so that the quantum evolution with respect to such clocks appears non-unitary.

Concerning evolution, the choice and corresponding notion of time is associated to the clock which we are employing. As indicated already by the deparameterizable system, the choice of clock within the effective treatment is implemented by gauge fixing: moments involving time (such as Δt) vanish (or take other prescribed and possibly complex values). The choice of time and clock is, thus, closely related to, and, in fact, nothing more than a gauge choice; we will refer to the choice as a *Zeitgeist*.

Applying such gauge conditions to non-global clocks, we encounter a striking feature: a consistent solution of the constraints and equations of motion requires complex-valued time. Interestingly, this novel feature can be implemented in a self-consistent manner and leads us to well-defined effective dynamics in a local time away from its classical extremal points. However, as we evolve closer towards these points with respect to the local clock, spreads and other moments diverge and become singular in the gauge corresponding to our choice of time [10]; this gauge and, thus, the choice of time becomes incompatible with the semiclassical expansion of moments. The apparent non-unitarity in a non-global time at the state level translates into the eventual breakdown of the corresponding gauge in the effective formalism. Fortunately, this ailment can be cured if we can find a new clock which is locally better-behaved where our original time variable becomes inadequate. Since the choice of time is nothing more than a gauge choice, we can switch clocks by a suitable gauge transformation, which in the effective treatment is generated by the constraint functions (such as (1)-(5)), in close analogy with classical constrained systems. Such transformations were, indeed, found explicitly in toy models and, hence, those systems could be evolved along semiclassical trajectories through the extremal points of local times, by "temporarily" switching to different clock functions [10]. In this way we attempt to reconstruct effectively a coherent physical state.

As regards the relational interpretation, we emphasize that each choice of a clock and the corresponding gauge comes with a different description of the system — its own Zeitgeist. Specifically, the moments of the kinematical operator used to measure time are fixed by the gauge conditions — only its expectation value remains free; the conjugate momentum of time

is entirely eliminated through the constraint functions. Therefore, in this description, neither operator could correspond to a physical variable, which could be meaningfully turned into a physical (relational) observable. Changing the clock and, thus, the notion of time, brings about a significant shift in perspective regarding the physical variables: the old clock and its conjugate momentum become physical in the new regime, while the newly chosen clock is relegated to the status of a parameter and its conjugate variable is altogether eliminated through constraints. Moreover, the accompanying gauge changes yield jumps of order $\hbar$ in physical correlations [10]. This has an important implication for (quantum) relational observables for non-deparameterizable systems, namely, one cannot construct relational observables which are valid for values of the relational clock near its turning points. In those regions we are forced to use a different clock and, therefore, to evolve a truly *different* set of "effective local relational observables". Trajectories for local relational observables in a new time, consequently, do not directly continue the preceding ones in the old time and leave a gap, although, nonetheless, consistently transporting along relational initial data. Time is of a local nature here and so is the relational concept of evolution.

Since we are dealing with a first-class constrained system, the concept of observables as gauge-invariant functions on phase space is still valid. What becomes limited in the absence of global internal times is the usual notion of evolving observables [3, 7, 11, 12, 13]. In particular, if local clocks have maximal or minimal values along classical trajectories, these extremal values typically vary from orbit to orbit. It may be the case for classical trajectories in some systems,[1] that sets of values (or even every value) of a given local clock lie beyond the maximal (or minimal) clock value allowed by the given classical orbit. Relational observables evolving in such a non-global clock are generally multi-valued and become complex beyond the extremal points, indicating that the system with given initial data will never reach such phase space points. Hence, the quantum version of a relational (Dirac) observable referring to this clock can, in principle, be a well-defined operator[2] on the physical Hilbert-space, but will, in general, yield complex expectation values in a physical inner product, thus, failing to be a self-adjoint operator on $\mathcal{H}_{\rm phys}$ (see also [7] on this issue). On the other hand, in a given Zeitgeist at the effective level one may formally compute expectation values of evolving observables, but a Zeitgeist is only rarely permanent. When it changes near a turning point of the local internal time, a different set of local relational observables is required. To be precise, by local relational observables at the effective level we mean correlations of expectation values and moments with the expectation value of the local clock variable, evaluated in its corresponding Zeitgeist. In this article, we call such local relational observables of the effective formalism computed with respect to a Zeitgeist "fashionables"; they constitute the complete physical information of interest about the system as long as the Zeitgeist remains intact, but may fall out of fashion when the Zeitgeist changes. The notion of a fashionable is, therefore, state-dependent, in contrast to the operator version of a quantum relational observable, as in different semiclassical states a given Zeitgeist is generally valid for different ranges of the associated local clock. Fashionables become invalid when the associated Zeitgeist / choice of time fails on approach to an extremal point of the local clock and, therefore, before the above mentioned issue of complex-valued correlations could set in. Fashionables, thus, reflect the local nature of relational quantum evolution and, by being state-dependent, are somewhat closer to physical interpretation.

[1] For instance, systems with closed orbits.

[2] Or multiple operators if the relational observable is multi-valued.

By analogy, we will also refer to expectation values of operators in Hilbert-space representations, obtained via local deparametrizations as discussed in section 3.2, as fashionables. It should be noted that in deparameterizable systems, where the Zeitgeist of the global clock is defined for its entire range, fashionables become globally valid and coincide with the expectation values of the standard operator versions of relational Dirac observables, obtained via deparametrization in the Dirac procedure.

3 Complex time

Several independent arguments indicate that internal time should be considered complex in non-deparameterizable systems. The full strength of this conclusion can be grasped only in the effective approach, which we will consider first, but it can be seen to arise also in Hilbert-space treatments.

3.1 Effective constraints and complex time

To be specific, we consider effective constraints for a relativistic particle in an arbitrary time-dependent potential $V(t,q)$. We will show that the imaginary contribution to $\langle\hat{t}\rangle$ is insensitive to the explicit form of V. Below, primes at the potential will refer to its partial q-derivatives and dots to its partial t-derivatives. We make use of the effective constraints

$$\begin{aligned} C &= \langle\hat{p}_t\rangle^2-\langle\hat{p}\rangle^2+(\Delta p_t)^2-(\Delta p)^2+V(\langle\hat{t}\rangle,\langle\hat{q}\rangle)+\tfrac{1}{2}\ddot{V}(\langle\hat{t}\rangle,\langle\hat{q}\rangle)(\Delta t)^2 \\ &\quad +\tfrac{1}{2}V''(\langle\hat{t}\rangle,\langle\hat{q}\rangle)(\Delta q)^2+\dot{V}'(\langle\hat{t}\rangle,\langle\hat{q}\rangle)\Delta(tq) && (7)\\ C_t &= 2\langle\hat{p}_t\rangle\Delta(tp_t)+i\hbar\langle\hat{p}_t\rangle-2\langle\hat{p}\rangle\Delta(tp)+\dot{V}(\langle\hat{t}\rangle,\langle\hat{q}\rangle)(\Delta t)^2+V'(\langle\hat{t}\rangle,\langle\hat{q}\rangle)\Delta(tq) && (8)\\ C_{p_t} &= 2\langle\hat{p}_t\rangle(\Delta p_t)^2-2\langle\hat{p}\rangle\Delta(p_tp)+\dot{V}(\langle\hat{t}\rangle,\langle\hat{q}\rangle)(\Delta(tp_t)-\tfrac{1}{2}i\hbar)+V'(\langle\hat{t}\rangle,\langle\hat{q}\rangle)\Delta(p_tq) && (9)\\ C_q &= 2\langle\hat{p}_t\rangle\Delta(p_tq)-2\langle\hat{p}\rangle\Delta(qp)-i\hbar\langle\hat{p}\rangle+\dot{V}(\langle\hat{t}\rangle,\langle\hat{q}\rangle)\Delta(tq)+V'(\langle\hat{t}\rangle,\langle\hat{q}\rangle)(\Delta q)^2 && (10)\\ C_p &= 2\langle\hat{p}_t\rangle\Delta(p_tp)-2\langle\hat{p}\rangle(\Delta p)^2+\dot{V}(\langle\hat{t}\rangle,\langle\hat{q}\rangle)\Delta(tp)+V'(\langle\hat{t}\rangle,\langle\hat{q}\rangle)(\Delta(qp)-\tfrac{1}{2}i\hbar)\,. && (11) \end{aligned}$$

To implement $\langle\hat{t}\rangle$ as local internal time, no t-moments should be present. The Zeitgeist $(\Delta t)^2=\Delta(tq)=\Delta(tp)=0$ should, thus, be a suitable way to fix the gauge. We first infer $\Delta(tp_t)=-\frac{1}{2}i\hbar$ from $C_t=0$. Then C_{p_t} implies

$$(\Delta p_t)^2=\frac{\langle\hat{p}\rangle}{\langle\hat{p}_t\rangle}\Delta(p_tp)+\frac{i\hbar\dot{V}(\langle\hat{t}\rangle,\langle\hat{q}\rangle)}{2\langle\hat{p}_t\rangle}-\frac{V'(\langle\hat{t}\rangle,\langle\hat{q}\rangle)}{2\langle\hat{p}_t\rangle}\Delta(p_tq)\,.$$

Eliminating $\Delta(p_tp)$ and $\Delta(p_tq)$ in the expression above using C_p and C_q, respectively, yields

$$(\Delta p_t)^2=\frac{\langle\hat{p}\rangle^2}{\langle\hat{p}_t\rangle^2}(\Delta p)^2+\frac{i\hbar\dot{V}(\langle\hat{t}\rangle,\langle\hat{q}\rangle)}{2\langle\hat{p}_t\rangle}+\frac{V'(\langle\hat{t}\rangle,\langle\hat{q}\rangle)^2}{4\langle\hat{p}_t\rangle^2}(\Delta q)^2-\frac{\langle\hat{p}\rangle V'(\langle\hat{t}\rangle,\langle\hat{q}\rangle)}{\langle\hat{p}_t\rangle^2}\Delta(qp)$$

and we finally obtain the alternative expression

$$\begin{aligned} C &= \langle\hat{p}_t\rangle^2-\langle\hat{p}\rangle^2+\frac{\langle\hat{p}\rangle^2-\langle\hat{p}_t\rangle^2}{\langle\hat{p}_t\rangle^2}(\Delta p)^2+\left(\frac{V''(\langle\hat{t}\rangle,\langle\hat{q}\rangle)}{2}+\frac{V'(\langle\hat{t}\rangle,\langle\hat{q}\rangle)^2}{4\langle\hat{p}_t\rangle^2}\right)(\Delta q)^2 \\ &\quad -\frac{\langle\hat{p}\rangle V'(\langle\hat{t}\rangle,\langle\hat{q}\rangle)}{\langle\hat{p}_t\rangle^2}\Delta(qp)+\frac{i\hbar\dot{V}(\langle\hat{t}\rangle,\langle\hat{q}\rangle)}{2\langle\hat{p}_t\rangle}+V(\langle\hat{t}\rangle,\langle\hat{q}\rangle) && (12) \end{aligned}$$

for the constraint $C = \langle\hat{C}\rangle$ on the space on which C_t, C_{p_t}, C_q and C_p are solved and the Zeitgeist is chosen as above. One may solve (12) for $\langle\hat{p}_t\rangle = H_Q$ as the quantum Hamiltonian for time t, and compute fashionables via the evolution it generates. Here, we are mainly concerned with properties of $\langle\hat{t}\rangle$.

In (12), terms not involving V and its derivatives should be real-valued: $\langle\hat{p}\rangle$ and $(\Delta p)^2$ (as well as $\langle\hat{q}\rangle$, $\Delta(qp)$ and (Δq)) are physical variables in this Zeitgeist which ought to be converted into (real-valued) fashionables by solving the equations of motion, and $\langle\hat{p}_t\rangle$ can be interpreted physically as the local energy value which is not conserved with a time-dependent potential but has a clear meaning. When the constraint is satisfied, we thus determine the imaginary part of $\langle\hat{t}\rangle$ from the equation

$$\mathrm{Im}\left(\left(\frac{V''(\langle\hat{t}\rangle,\langle\hat{q}\rangle)}{2}+\frac{V'(\langle\hat{t}\rangle,\langle\hat{q}\rangle)^2}{4\langle\hat{p}_t\rangle^2}\right)(\Delta q)^2-\frac{\langle\hat{p}\rangle V'(\langle\hat{t}\rangle,\langle\hat{q}\rangle)}{\langle\hat{p}_t\rangle^2}\Delta(qp)+\frac{i\hbar\dot{V}(\langle\hat{t}\rangle,\langle\hat{q}\rangle)}{2\langle\hat{p}_t\rangle}+V(\langle\hat{t}\rangle,\langle\hat{q}\rangle)\right)=0\,, \tag{13}$$

which, in general, can be difficult to solve and does not seem to give rise to a simple, universal, potential-independent imaginary part of $\langle\hat{t}\rangle$. For semiclassical states, however, to which this approximation of effective constraints refers anyway, we Taylor expand the potential in the imaginary term, expected to be of the order $\hbar$,

$$V(\langle\hat{t}\rangle,\langle\hat{q}\rangle)=V(\mathrm{Re}\,\langle\hat{t}\rangle+i\,\mathrm{Im}\,\langle\hat{t}\rangle,\langle\hat{q}\rangle)=V(\mathrm{Re}\,\langle\hat{t}\rangle,\langle\hat{q}\rangle)+i\,\mathrm{Im}\,\langle\hat{t}\rangle\,\dot{V}(\mathrm{Re}\,\langle\hat{t}\rangle,\langle\hat{q}\rangle)+O((\mathrm{Im}\,\langle\hat{t}\rangle)^2)\,,$$

while derivatives of the potential are expanded in an identical manner. To order $\hbar$, only the last two terms of (13) possess an imaginary part. The imaginary contribution to C, which is constrained to vanish independently of the real part of C, is then given by $\frac{1}{2}i\hbar\,\dot{V}(\mathrm{Re}\,\langle\hat{t}\rangle,\langle\hat{q}\rangle)/\langle\hat{p}_t\rangle+i\,\dot{V}(\mathrm{Re}\,\langle\hat{t}\rangle,\langle\hat{q}\rangle)\,\mathrm{Im}\,\langle\hat{t}\rangle+O(\hbar^{3/2})=0$. Thus, it immediately follows that

$$\mathrm{Im}\,\langle\hat{t}\rangle=-\frac{\hbar}{2\langle\hat{p}_t\rangle}\,. \tag{14}$$

As the derivation shows, this imaginary contribution to time is a universal result, independent of the potential. In particular, time must be complex, although with its imaginary part determined by phase-space variables, it still provides a 1-dimensional flow.

There are old and well-known arguments in quantum mechanics saying that time cannot be a self-adjoint operator, for it would be conjugate to an energy operator bounded from below for stable systems. Since a self-adjoint time operator would generate unitary shifts of energy by arbitrary values, a contradiction to the lower bound would be obtained. The result obtained here looks similar at first sight — a non-self-adjoint time operator could, certainly, lead to complex time-expectation values — but it is more general. In the present example, we are using an arbitrary potential which does not necessarily provide a lower bound for energy. The usual arguments about time operators, thus, do not apply; instead, our conclusions are drawn directly from the fact that we are dealing with a time-dependent potential in a non-deparameterizable system. (For time-independent potentials, $\langle\hat{t}\rangle$ does not appear in the effective constraints and can consistently be chosen real. The time dependence is, therefore, crucial for the present discussion.)

The value of the imaginary part of time is directly related to the Zeitgeist as we have used it in the derivation. Were we to change time, which is often required in non-deparameterizable systems which lack a global internal time, we change the Zeitgeist. Accordingly, the accompanying gauge transformation must transfer the imaginary contribution from the old time to

the new one since the above argument will also hold for the new clock in its corresponding gauge. This can also be demonstrated explicitly in toy models [10].

By implementing changes of time as mere gauge transformations in a first-class system of constraints, we solve the multiple-choice aspect of the problem of time. Since the effective approach works at an algebraic level, rather than directly with Hilbert-space representations, the Hilbert-space aspect of the problem of time is bypassed as well. In fact, one may view different gauge choices of the effective constrained system as different representations that would normally be realized at the Hilbert-space level. The only price to pay is that we must deal with complex time, which may be unfamiliar but does not pose any additional difficulties. In the remainder of this section we will show that the same value of the imaginary part can be seen to arise not only in the effective approach.

3.2 Schrödinger regime for relativistic systems

In a Dirac-type quantization the main difficulty is usually to determine a physical inner product with physical evolution, for which no systematic treatment exists in the case of non-deparameterizable systems.[3] Moreover, it seems difficult to shed light on the origin of imaginary contributions to time from this perspective since there is normally no clock operator defined on the physical Hilbert space, whose physical expectation value one could compute. Instead of the physical inner product associated with the relativistic system, we will consider a Schrödinger equation which linearizes the relativistic equation in the momentum of internal time.

A relativistic constraint equation of the form

$$\left(\hat{p}_t^2 - \hat{H}^2(\hat{t},\hat{q},\hat{p})\right)\psi(q,t) = 0\,, \tag{15}$$

where $\hat{H}^2$ is a positive operator at least on a subset of states, is in general not equivalent to the Schrödinger equation

$$\left(-i\hbar\partial_t + \hat{H}(t,\hat{q},\hat{p})\right)\psi(q,t) = 0\,, \tag{16}$$

as it would be in the case of a time-independent Hamiltonian for positive-frequency solutions. Solutions to (16) rather satisfy the relativistic version

$$-\hbar^2\partial_t^2\psi = \hat{H}^2\psi + i\hbar\partial_t\hat{H}\psi \tag{17}$$

of the constraint.

In this comparison, we implicitly assume, however, that t refers to the same time variable in both cases, and, in particular, that it always takes real values. In (16), t is a time parameter not associated with any operator and it would be difficult to justify it taking complex values. In (15), however, t is an internal variable and quantized; its real-valuedness depends on the adjointness properties of $\hat{t}$, a question that brings us back to the physical inner product. While a physical inner product is difficult to find in such non-deparameterizable situations,

[3] Generalizations of Klein–Gordon type physical inner products have been suggested based on the notion of asymptotic positive-frequency solutions [14]. Another method is based on spectral decomposition [15]. In those cases, defining physical evolution, especially through turning points of local internal times, remains a challenge.

one can nevertheless argue that an imaginary contribution to $\langle\hat{t}\rangle$ in the system described by (15) is required in order to provide equivalence with (16).

To do so, we rewrite the right-hand side of (17) and require

$$\hat{H}^2(\tau,\hat{q},\hat{p}) + i\hbar\partial_\tau \hat{H}(\tau,\hat{q},\hat{p}) = \hat{H}^2(\hat{t},\hat{q},\hat{p})\,, \tag{18}$$

where for distinction we have renamed the parameter t of the Schrödinger equation by τ. Here, $\hat{t}$ is the clock operator in the relativistic system, which may not be self-adjoint, and $\tau\in\mathbb{R}$ is to be related to $\langle\hat{t}\rangle$ in some way so as to achieve equivalence with the Schrödinger equation. One can already see from this equation that imaginary contributions to $\langle\hat{t}\rangle$ will be required if the left-hand side is interpreted as some kind of expansion of the right-hand side to order $\hbar$. In addition to deriving the imaginary contribution, it remains to be shown that $-\hbar^2\partial_\tau^2$ can be interpreted as $\hat{p}_t^2$, i.e., in terms of the momentum conjugate to the new operator $\hat{t}$, at least on solutions to (16). If this is the case, (17) turns into (15) with $\hat{t}$ related to τ.

To perform the derivations in the semiclassical approximation, as sufficient for a comparison with our effective equations, we compute expectation values of $\hat{H}^2(\hat{t},\hat{q},\hat{p})$ in solutions to (16) assuming the standard Schrödinger inner product up to order $\hbar$. Then we have $\langle\hat{t}\rangle^2 = \langle\hat{t}^2\rangle$, $\langle\hat{t}\hat{q}\rangle = \langle\hat{t}\rangle\langle\hat{q}\rangle$ and $\langle\hat{t}\hat{p}\rangle = \langle\hat{t}\rangle\langle\hat{p}\rangle$, just as we have it for the Zeitgeist associated to $\hat{t}$ of the effective approach with $(\Delta t)^2 = \Delta(tq) = \Delta(tp) = 0$; thus,

$$\langle\hat{H}^2(\hat{t},\hat{q},\hat{p})\rangle = \langle\hat{H}^2(\langle\hat{t}\rangle,\hat{q},\hat{p})\rangle + o(\hbar^{3/2})\,. \tag{19}$$

(Even higher-order moments involving t can be expected to vanish, but equalities here are required only up to order $o(\hbar^{3/2})$.)

We now postulate the relation between $t=\langle\hat{t}\rangle$ and the Schrödinger time $\tau\in\mathbb{R}$ as $t = \tau + i\hbar T$ with T to be determined. Continuing to expand the right-hand side of (19), we have

$$\begin{aligned}\langle\hat{H}^2(t,\hat{q},\hat{p})\rangle &= \langle\hat{H}^2(\tau,\hat{q},\hat{p})\rangle + 2i\hbar T\langle\hat{H}(\tau,\hat{q},\hat{p})\partial_\tau\hat{H}(\tau,\hat{q},\hat{p})\rangle + o(\hbar^{3/2})\\ &= \langle\hat{H}^2(\tau,\hat{q},\hat{p})\rangle + 2i\hbar T\langle\hat{H}(\tau,\hat{q},\hat{p})\rangle\langle\partial_\tau\hat{H}(\tau,\hat{q},\hat{p})\rangle + o(\hbar^{3/2})\,.\end{aligned} \tag{20}$$

Combining (19) and (20), we obtain (18) in terms of expectation values if $T = \frac{1}{2\langle\hat{H}\rangle} = \frac{1}{2\langle i\hbar\partial_\tau\rangle}$, the latter equality on solutions of (16).

By construction, recalling (17), we then have

$$\langle\hat{H}^2(\hat{t},\hat{q},\hat{p})\rangle = \langle -\hbar^2\partial_\tau^2\rangle \tag{21}$$

to semiclassical order. For partial time derivatives the imaginary contribution to $\langle\hat{t}\rangle$ does not matter, and we may replace ∂_τ by ∂_t:

$$\langle\hat{H}^2(\hat{t},\hat{q},\hat{p})\rangle = \langle -\hbar^2\partial_t^2\rangle = \langle\hat{p}_t^2\rangle\,. \tag{22}$$

To semiclassical order solutions to (16) satisfy a relativistic constraint equation if we interpret the expectation value of the time operator in the latter to be complex with the same imaginary contribution $\mathrm{Im}\,\langle\hat{t}\rangle = -\frac{\hbar}{2\langle\hat{p}_t\rangle}$, as seen in the effective approach (14).

In terms of operators at a kinematical level, we can identify

$$\hat{t} = \hat{\tau} - \frac{i\hbar}{2}\widehat{p_\tau^{-1}} \tag{23}$$

(for states lying outside the zero-eigenspace of $\hat{p}_\tau$, i.e. outside "turning points"). With this identification, we can further justify replacing ∂_τ by ∂_t: thanks to $[\hat{t},\hat{p}_\tau] = i\hbar$, the momenta

$\hat{p}_t = \hat{p}_\tau$ agree. The Schrödinger and relativistic formulations provide different representations of the dynamics with different Hilbert spaces. In the representation-independent effective formulation we have the gauge-fixed constraint (12) as the Schrödinger analog, and the non-gauge-fixed (7) as the relativistic analog. In fact, in the toy models studied in [10], the semiclassical dynamics produced by locally deparametrizing the relativistic constraint with a Schrödinger equation matches precisely the effective dynamics derived using the corresponding Zeitgeist.

3.3 Complex time in deparameterizable systems

An imaginary contribution to time can be seen also from the well-known physical inner product formulas available for deparameterizable systems. An imaginary contribution is not required in those systems from an effective procedure or for a Schrödinger regime, but one can still see how it may arise naturally.

We consider the free relativistic particle in 1+1 dimensions, described by a complex-valued scalar wavefunction of two variables, $\psi(x_0, x_1)$, subject to the constraint

$$\left(-\hbar^2 \frac{\partial^2}{\partial x_0^2} + \hbar^2 \frac{\partial^2}{\partial x_1^2} - m^2\right) \psi(x_0, x_1) = 0\,. \tag{24}$$

General solutions have the form

$$\psi_{\rm phys}(x_0, x_1) = \int_{-\infty}^{\infty} \left(f_+(k) e^{i\hbar^{-1}(kx_1 - \epsilon_k x_0)} + f_-(k) e^{i\hbar^{-1}(kx_1 + \epsilon_k x_0)}\right) {\rm d}x_1\,, \tag{25}$$

where $\epsilon_k = \sqrt{k^2 + m^2}$. Solutions in this general form automatically split into positive-frequency and negative-frequency components, a split which is important for constructing the physical Hilbert space (see, e.g., [16]). On positive-frequency solutions, the physical inner product is

$$(\phi, \psi) := i\hbar \int_{-\infty}^{\infty} \left(\bar{\phi}(x_0, x_1) \frac{\partial}{\partial x_0} \psi(x_0, x_1) - \left(\frac{\partial}{\partial x_0} \bar{\phi}(x_0, x_1)\right) \psi(x_0, x_1)\right) {\rm d}x_1 \bigg|_{x_0 = t} \tag{26}$$

with an extra minus sign for negative-frequency solutions, while negative-frequency and positive-frequency solutions are mutually orthogonal. When evaluated on solutions to (24), the integration is independent of the value of t.

We are interested in an analog of a time operator, which cannot be an observable. Thus, it does not preserve the space of solutions, but we can still compute expectation values using (26) as a bilinear form on the kinematical Hilbert space. For non-observable operators, the expectation values will be time dependent just as we need it for t itself. For example, for $\hat{q} = x_1$ and $\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x_1}$ the time-dependent expectation values correspond precisely to the usual dynamics of the free relativistic particle. Applying this procedure to the time operator, it then becomes apparent that the expectation value of $\hat{t} = x_0$ (on positive-frequency solutions ϕ^+, to be specific) is not only time dependent but complex:

$$\begin{aligned}
(\phi^+, \hat{t}\phi^+) &= i\hbar \int_{-\infty}^{\infty} \left(\bar{\phi}^+(x_0, x_1) \frac{\partial}{\partial x_0} \left(x_0 \phi^+(x_0, x_1)\right) - \left(\frac{\partial}{\partial x_0} \bar{\phi}^+(x_0, x_1)\right) x_0 \phi^+(x_0, x_1)\right) {\rm d}x_1 \bigg|_{x_0 = t} \\
&= i\hbar \int \bar{\phi}^+ \phi^+ {\rm d}x_1 \bigg|_{x_0 = t} + i\hbar \int x_0 \left(\bar{\phi}^+ \frac{\partial}{\partial x_0} \phi^+ - \left(\frac{\partial}{\partial x_0} \bar{\phi}^+\right) \phi^+\right) {\rm d}x_1 \bigg|_{x_0 = t} \\
&= i\hbar \left\langle \widehat{\frac{1}{2\epsilon_k}} \right\rangle + t = t - \frac{i\hbar}{2} \left\langle \widehat{\frac{1}{p_t}} \right\rangle.
\end{aligned}$$

(Note that the action of $\hat{p}_t$ on positive-frequency solutions is equivalent to multiplication by $-\epsilon_k$ in momentum space.) Again, to order $\hbar$ the imaginary part of $\langle\hat{t}\rangle$ is in agreement with the one seen in the effective approach.

For non-deparameterizable systems we do not have an explicit physical inner product at our disposal, but we can argue heuristically that the time expectation value should be complex. We assume a constraint of the form

$$\left(-\hbar^2\frac{\partial^2}{\partial x_0^2}-\hat{H}^2(\hat{q},\hat{p},\hat{t})\right)\psi(x_0,x_1)=0\,, \tag{27}$$

where $\hat{H}^2$ contains no time derivatives (and thus commutes with $\hat{t}=x_0$) but may be time-dependent. Solving a second-order partial differential equation as a constraint, we expect the physical inner product to depend on both $\psi(x,t)$ and $\frac{\partial}{\partial t}\psi(x,t)$. Indeed, it can be shown that (26) is conserved in time for the solutions of any constraint of the form given in (27), so long as $\hat{H}^2$ is self-adjoint as an operator on $L^2(\mathbb{R},\mathrm{d}x)$, for each value taken by t. However, the expression is not positive definite in general. It is not difficult to see, that an inner product involving both $\psi(x,t)$ and $\frac{\partial}{\partial t}\psi(x,t)$ will likely assign a complex expectation value to $\langle\hat{t}\rangle$, since $\hat{t}$ as a kinematical operator maps $\psi(x_0,x_1)$ to $x_0\psi(x_0,x_1)$, and

$$\hat{t}\left(\frac{\partial}{\partial x_0}\psi(x_0,x_1)\right):=\frac{\partial}{\partial x_0}(x_0\psi(x_0,x_1))=\left(t\hat{\mathbb{1}}+i\hbar\hat{p}_t^{-1}\right)\frac{\partial}{\partial x_0}\psi(x_0,x_1)\,. \tag{28}$$

3.4 What time is it?

Confronted with a complex time, we are charged with the task of elucidating the notion of such a "vector time" with its apparently two separate degrees of freedom. The particular form of (14) directly implies that the imaginary contribution to the clock function is a constant of motion in the absence of a time-dependent potential while it becomes dynamical in the presence thereof. A constant imaginary contribution can be disregarded altogether for relational evolution and is also not required in order to solve the constraints. (15) and (16) are automatically equivalent in this case; in addition, time does not appear in the effective constraint functions and may, therefore, be chosen real. A dynamical imaginary contribution, on the other hand, can, certainly, not be neglected in the constraints and when discussing relational evolution. Note that a non-global clock necessarily implies a time-dependent potential, but a time-dependent potential does not necessarily imply a non-global clock. For instance, in a relativistic system with a constraint $C=p_t^2-H^2(t,q,p)$, where $H^2>0\;\forall\,t$, the variable t will be a global clock. The dynamical imaginary contribution is, thus, more general than a mere consequence of non-unitarity, but becomes most significant where the momentum conjugate to the clock becomes very small and, accordingly, plays a more pivotal role for non-global internal times.

This, in fact, also leads us to the discussion of the quality of the relational clock. For instance, in [17] it is advocated that fundamental uncertainties for relational observables could arise as a result of using a dynamical variable as clock which should be disturbed during the measurement of a complete relational observable. Different clock variables will lead to different resolutions for relational observables and fundamental uncertainties could result in general. In [18], Poisson brackets of relational observables are considered from which the uncertainties will follow in the quantum theory. The inverse kinetic energy of the clock appears in these Poisson-brackets and it is argued that the clock is better, the greater

its (kinetic) energy,[4] corresponding to the intuition that the faster the clock, the finer its time-resolution. In agreement with this, it is found in [20] that the quantum notion of a time-of-arrival of a particle, which in a relational context could be employed as a clock, is limited by an inherent uncertainty which is inversely proportional to the kinetic energy of the (clock-)particle. This discussion is in close parallel to the relations found in this article. The particular imaginary contribution to the clock is smaller, the larger its kinetic energy, which is compatible with the fact that the local clock is better behaved away from its turning point where quantum uncertainties limit its applicability (see also the following section on this issue).

As regards relational evolution, we opt to use $\mathrm{Re}[\langle\hat{t}\rangle]$ rather than $\mathrm{Im}[\langle\hat{t}\rangle]$ as the physical time for several reasons: 1) in the classical limit the imaginary part of $\langle\hat{t}\rangle$ vanishes and it is, indeed, the real part of $\langle\hat{t}\rangle$ that matches the classical time; 2) thanks to (14), away from the extremal points, the imaginary part of $\langle\hat{t}\rangle$ is small and approximately constant, thus, providing poor parametrization of dynamics; 3) in the Schrödinger regime which linearizes the relativistic constraint, the time parameter refers precisely to the real part of $\langle\hat{t}\rangle$ (this is, in fact, related to the previous point as the Schrödinger regime is only applicable away from the extrema); 4) the explicit inner product that reproduces $\mathrm{Im}[\langle\hat{t}\rangle]$ in the case of a free relativistic particle is based on integrating at a fixed value of (parameter) t equal to precisely the real part of the corresponding expectation value, and 5) the dynamical imaginary term can fail to be monotonic where the real part operates as an appropriate local clock.

4 Time in a highly quantum state

> *About the only time we get any let-up from this time control is in the fog; then time doesn't mean anything. It's lost in the fog, like everything else.*
>
> Ken Kesey: One flew over the Cuckoo's nest

Although the specific equations developed here apply only to semiclassical regimes, general properties of effective constraints allow us to shed some light on the issue of (non-global) time in general quantum states. The differences in relational evolution between the classical and quantum theory merely result, as usual, from the quantum uncertainties, however, the latter have more severe repercussions in the absence of a global clock which at the classical level, in fact, does not constitute a deep conceptual problem. As always with highly quantum states, intuition becomes rather foggy; but effective techniques, by being closer in spirit to the classical formulation than state representations, can provide valuable input. The role of time in a highly quantum state is a question of considerable fundamental interest, and it has been discussed before. Given the difficult nature of this problem, possible answers put forward so far have remained rather vague. Proposals derived from the effective constraints in this paper, expanded semiclassically, will be no less vague. But the viewpoint they provide is new, we believe, and the light they shed worth shining.

Recall that at the effective level and to semiclassical order, a variable can assume the role of a suitable clock wherever its corresponding Zeitgeist, which fixes all but one effective gauge flow, is consistent with the assumed hierarchy and fall-off properties of moments in orders of $\hbar$ as described in section 2. The choice of a Zeitgeist such as (6), which projects the clock variable

[4] Certainly, large energies are a delicate issue in general relativity, essentially due to black hole forming, but see [17, 19] and references therein on this issue in the context of fundamental limits on physical clocks.

to merely a "classical" parameter by setting its fluctuations to zero,[5] can be interpreted as the effective analogue of choosing a constant clock-time slicing in a deparametrization at the Hilbert-space level which also renders the clock variable essentially "classical", regardless of whether the state is semiclassical or not.[6] In particular, it is really the choice of the clock variable which determines how quantum spreads are measured. Consider, e.g., the deparametrizable example of the free Newtonian particle governed by the constraint $C = p_t + p^2$. Here both t and q are good global clocks and in the quantum theory the physical state solving the quantum version of the constraint will be a priori "there at once" and infinitely spread in both t and q directions. However, irrespective of how highly quantum the state, we can deparametrize in either t or q by choosing a corresponding slicing on which the physical inner product will be defined. It is this choice of the clock variable which will collapse it to the role of a "classical" parameter and determine how the spreads of the state are measured, i.e., in this case whether they are measured on a $t = \text{const}$ or on a $q = \text{const}$ slice. The clock variable which appears "classical" in its corresponding slicing might itself appear "highly quantum" in the slicing corresponding to the other clock choice.

Consider now a system which has no global clock and whose local clocks have maximal or minimal values along the classical trajectories. As a prototype of a highly quantum state, consider a superposition of two or more semiclassical states. For each classical trajectory, extremal values of a given non-global time variable are, in general, different, and, therefore, for each of the corresponding semiclassical states the gauge associated to the clock choice breaks down at different instants of relational time. For a superposition of two such states it follows that the region, where a given time variable is invalid, is larger than for the individual states. As we superimpose more and more semiclassical states to obtain a highly quantum solution to the constraint, it is possible, that, e.g., for systems with closed classical orbits, no regions remain where a given local time variable can be used as a clock. The more quantum the state, the more effective variables, i.e. higher moments, and quantum constraint functions we have to take into account. In such situations it also becomes clear that an analogue of a gauge associated to the clock, such as the Zeitgeist (6) which forces the clock into the role of a "classical" parameter, becomes less and less consistent when the quantum nature of the clock is no longer negligible. In particular, the fluctuations associated to the momentum conjugate to the clock may become large as a consequence of superposition of positive and negative values of the momentum, or, in other words, of opposite time directions. Note that our construction of the effective dynamics using local times does not require that the fluctuations of *all* degrees of freedom can consistently be set to zero or maintained small, but only of the ones that we want to appoint as clocks. In addition and related to this, the clock should possess sufficient

[5]It should be noted that due to the complex-valuedness of unphysical moments, generalized uncertainty relations are respected even when certain fluctuations are zero. For instance, in section 2.1 it was discussed that the Zeitgeist (6) actually leads to a saturation of the (generalized) uncertainty relation for the clock variable, a property often associated with a strict form of semiclassicality.

[6]The "classicality" of an internal time variable may be counterintuitive because time, being canonically conjugate to a constraint, must be spread out over large domains even if it is valid only locally. A state in which time behaves semiclassically, by contrast, may be expected to have a sharply peaked behavior along the time direction such that time seems unable to progress much. The apparent contradiction in the notion of semiclassical time is resolved by noting that a wave function solving the constraint is indeed spread along the time direction, but that semiclassicality must physically be determined through properties of the Dirac observables. States are spread out kinematically, but time is not an observable and thus lacks obvious measures for semiclassicality. (The semiclassicality of other variables, by contrast, must be derived by solving the constraints and is not automatically guaranteed.)

kinetic energy, otherwise its resolution is poor and its imaginary contribution becomes large. If the clock ticks very slowly, other variables may change significantly in a short interval of clock time such that their evolution cannot be properly resolved and fluctuations appear large. Thus, if a highly quantum state has any degree of freedom that admits a consistent "projection to a classical parameter" and possesses a sufficiently large kinetic energy, there is a hope that effective dynamics can be defined.

Other methods for defining local time evolution discussed here, fare no better in a highly quantum state. In general, such a state admits superpositions of time directions, i.e. of positive and negative frequencies associated to the spectrum of the momentum conjugate to the clock. This superposition becomes an issue already for semiclassical states in the turning region of the local clock, where its conjugate momentum approaches zero, so that both positive and negative frequencies become relevant. This issue worsens if the spreads are so large that the segments of the wave function before and after the turning region start overlapping. The local Schrödinger regime of section 3.2 relies on using a square-root operator, which can only be defined on positive or negative frequency solutions separately. Mixing of the frequencies has the consequence that we can no longer locally deparametrize in the clock which would yield a local Schrödinger type evolution in only one given time direction generated by the corresponding Hamiltonian; equivalence of this regime with the full relativistic constraint, as discussed in section 3.2, cannot be established anymore and only the latter is valid. Additionally, in the presence of mixed time directions, simple inner products based on evaluation at constant clock-time surfaces seem to be inapplicable and, as a consequence, it is difficult to see how one could define unitary time evolution in those cases. As a simple (deparametrizable) example consider once again the free relativistic particle subject to the constraint equation (24) of section 3.3. This equation is hyperbolic and the initial value problem (IVP) is a priori well-posed, but a general solution (25) will include both positive and negative frequencies. Consequently, the constant-time inner product given by (26) fails to be positive-definite and cannot on its own provide us with a physically meaningful unitary interpretation of the evolution. Only if we impose the further restriction of only considering, e.g., positive frequency modes, do we have a positive-definite physical inner product and a physically meaningful solution to the IVP. The latter is owed to the fact that restriction to positive frequencies is tantamount to imposing a (in this case forward pointing) time direction.[7] It seems hardly imaginable that, in more general scenarios with frequency mixings, inner products relying on constant clock-time surfaces are meaningful. These are usually also closely linked to an — at least local — unitary evolution of initial data in some clock time, generated by some suitable Hamiltonian. But in a highly quantum state of a system with no global time even local unitary evolution becomes meaningless close to the turning region where frequency mixing is significant — apart from the fact that positive and negative frequencies require two separate Hamiltonians for evolution. A physical inner product based on more general boundaries or on the entire configuration space is in general required to cope with such highly quantum scenarios.

Here, however, we rely on local deparametrizations and, therefore, on disentangling frequencies; at the state level we would like to pose some IVP at an instant of relational time and

[7] Also in the classical treatment of relativistic systems, where the square of the momentum conjugate to the clock appears in the constraint, one is required to specify the time direction in order to formulate a relational IVP. Namely, given the initial data of the other variables at the initial value of the clock, one can only solve the constraint up to sign for the momentum conjugate to the clock. One is forced to choose a sign which then determines the time direction.

at least locally evolve this initial data unitarily, at the effective level we would like to impose a gauge such as (6) and formulate an IVP at a given clock value only on a segment of the semiclassical orbit, outside the region where a local clock breaks down, and then evolve data through this region (using a different local clock). As a result, this relational concept seems to be of a merely semiclassical nature and breaks down earlier than the classical evolution in a given clock. The more quantum the state, the earlier the apparent non-unitarity sets in and the earlier the relational evolution becomes meaningless. For sufficiently semiclassical states it is still possible to switch the clock before non-unitarity sets in,[8] which amounts to a gauge change in the effective framework and a change of constant clock-time slicing in local deparametrizations at the state level. But for highly quantum states this notion of evolution seems to disappear together with the notion of relational time; if there is no valid Zeitgeist at the effective level there can also be no fashionables. This, in fact, is compatible with the breakdown of relational observables close to turning points in the context of reduced phase space quantization discussed in [7].

The imaginary contribution to time, similarly, is related to local deparametrizations and constant clock-time slicings; at the effective level it appears in the gauge associated to the clock choice and in sections 3.2 and 3.3 it showed up in expectation values evaluated in inner products based on constant time slicings. This complex time might, in fact, obtain further contributions as we go to higher orders, but, in general, for an arbitrary quantum state when local deparametrizations and disentanglement of frequencies are no longer possible, complex time will disappear together with the notion of relational evolution.

We emphasize that the effects considered here are the result of imposing a relational interpretation on and attempting a local reconstruction of physical states of systems without global clocks and without the usual time structure via local deparametrizations. The apparent non-unitarity and any decoherence associated to this are, therefore, a mere result of this interpretation. A priori, the system may simply lack the standard notion of time-evolution — and, therefore, of non-unitarity — altogether.

An arbitrary quantum state will be governed by the full relativistic constraint and any expectation values of Dirac observables are to be taken with respect to the physical inner product, which in general cannot be constructed by evaluating data on a constant clock-time surface. From the point of view of partial differential equations, it is hard to see how time-evolution could emerge in general. A general constraint equation, may not provide a well-posed IVP in any variable at all. But even if, for a given constraint, the IVP is well-posed on some constant-time surface, its solution could turn out to be non-unitary or even non-time-reversible, in the sense that the data at some later value of relational time is compatible with a multitude of initial data at the initial surface. Furthermore, such an initial surface of constant clock time will, in general, intersect the flow generated by the classical constraint more than once. Consequently, assigning initial data on the whole of such a surface lacks clear physical interpretation as an IVP in the standard sense. From a Hilbert-space point of view, it is not clear how to interpret a general state or distribution (which is after all what one obtains by solving the constraint) with arbitrary shape/fall off properties as an at least locally unitary evolution of some sort. Of course, this does not entirely preclude that there may be a more fundamental way to define dynamics with respect to some more basic notion of time which goes beyond the issue of superposition of time directions and reduces to mere (fuzzy) correlations in a reduced phase space or even Dirac quantization. However, this

[8] In this sense admitting unitary evolution through the turning point of the clock.

remains questionable and even in the standard relational procedure constructions of quantum relational observables in the literature remain generally tentative for systems without global clocks and have otherwise only been successfully completed in the deparametrizable case. In contrast to this, the advantage of the effective approach is that it naturally gives rise to the notion of fashionables semiclassically and offers an outlook to more quantum regimes, suggesting that the nature of time changes as one motions from classical behavior to more highly quantum states.

5 Discussion

We have applied the effective procedure of dealing with quantum constraints to non-deparameterizable systems. Traditional procedures to deal with physical evolution are difficult to apply for those systems, but the effective approach is very feasible, at least for semiclassical questions which are often of most interest in those aspects of quantum gravity or cosmology that have at least a slight chance of being potentially observable. Within the same approach, not only solving the constraints but also finding relational observables benefits from strong simplifications compared to calculations for operators and states in a Hilbert-space representation. Computing explicit observables may still be complicated, but no conceptual problems occur and numerical tools can easily be implemented.

In particular, many of the facets of the problem of time are evaded by the possibility of patching together local internal times in quantum systems, just as one could do it for classical systems. Physicality conditions for observables are implemented just by reality conditions; no integral representation of an inner product need be constructed. Quantizing non-deparameterizable systems often fails already at the step of constructing a physical inner product, a strong handicap for canonical quantum gravity which is completely avoided by the effective techniques. Even if one were to know a physical inner product, finding sufficiently many quantum observables is often a problem. This step as well is simplified in the effective formalism which is treated in a classical manner and is also more amenable to numerical implementations. With the methods described here, we, thus, expect that much headway can be made in evaluating quantum-gravity theories and models in a practical way. Obviously, the effective setting remains to be extended, most importantly to quantum field theories, before it becomes applicable to full quantum gravity in semiclassical regimes. Promisingly, in formulations such as loop quantum gravity one can replace the continuous field theories of gravity by systems of finitely many degrees of freedom in compact regions, for instance by focussing attention on suitable classes of spin-network states which still capture the full amount of degrees of freedom. Therefore, crucial issues of quantum field theory should not be expected to make the effective techniques for constrained systems inapplicable in the context of gravity in inhibiting ways.

The main advantage for non-deparameterizable systems is that local internal times can be made sense of in the first place, and patched together by simple gauge changes. With changes of complex local internal times shown to be fully consistent, patching local internal times becomes a valid procedure, overcoming the (global and multiple choice) problem of time. For any concrete system, one will notice the breakdown of one's initial choice of internal time when its momentum becomes small, approaching a turning point of time. Before the momentum takes on too small values, which would endanger the validity of the semiclassical approximation, one can transform to a new internal time. No sharp instant can be provided

for when the change of time should be performed, but it is not relevant as long as the change of time is performed well before the breakdown of the initial choice of time [10].

The most striking feature of non-global clocks is their necessary complex-valuedness which can be elucidated by general arguments. Regarding evolution, we advocate in this article to appoint only the real part as relational time. Local internal times, furthermore, allow one to extend the notion of quantum relational observables to non-deparameterizable systems. Fashionables such as $\langle\hat{q}\rangle(\langle\hat{t}\rangle)$ are state-dependent and result even if $\langle\hat{t}\rangle$ is not used as internal time throughout the whole evolution. However, local internal times imply several subtleties, and for this reason the notion of fashionables is more general than the one of relational observables developed mainly with deparameterizable systems in mind. For instance, we are obliged to perform a change of time and associated gauge prior to the turning point of the clock, resulting in (order $\hbar$) discontinuities in correlations and the necessity to evolve a truly *different* set of fashionables in the new Zeitgeist. The relational concept is of a genuinely local nature here.

By sidestepping the Hilbert space problem, the effective approach also offers a vague outlook on the nature of time in highly quantum states, indicating that the usual concept of evolution in a relational time disappears in highly quantum regimes of systems devoid of global clocks.

For actual physical predictions of a theoretical framework, relational Dirac observables are not required to be defined or known for all values of relational time. Observations refer only to finite ranges of time, and so for predictions relational observables need be known only for finite ranges of relational time. These finite ranges may not even be contiguous. For instance, one may be interested in the change of an observable as the variable used as internal time moves through a turning point. (A cosmological application may be the evolution through a bounce.) Differences in Zeitgeist do not pose problems for these questions; one could simply disregard any gaps in the complete relational evolution as well as in differences of gauge for intermediate periods bringing one through the turning point of an internal time of interest. For evolution before and after the turning point the same local internal time can be used, removing potential interpretational difficulties that could arise from the use of different choices of Zeitgeist.

For further discussion of the issues raised in this article and concrete examples, we refer the interested reader to [10].

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